

Regularities of Twin, Triplet and Multiplet Prime Numbers

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Abstract

Classifications of twin primes are established and then applied to triplets that generalize to all higher multiplets. Mersenne and Fermat twins and triplets are treated in this framework. Regular prime number multiplets are related to quadratic and cubic prime number generating polynomials.

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1 Introduction

There exist extensive tables of twin, triplet and quartet primes. There are no such systematic analyses of higher multiplets. Here we outline a more systematic analysis of generalized twin primes, triplets and regular multiplets that are connected with prime producing quadratic and cubic polynomials. We follow standard practice ignoring as trivial the prime pairs $(2, p)$ of odd distance $p - 2$ with p any odd prime.

2 Classifications of twin and triplet primes

Definition 2.1. A triplet $p_i, p_m = p_i + 2d_1, p_f = p_m + 2d_2$ of odd prime numbers with $p_i < p_m < p_f$ is called a *generalized triplet*.

Each triplet consists of three *generalized twin primes* $(p_i, p_m), (p_m, p_f), (p_i, p_f)$. Empirical laws governing triplets therefore are intimately tied to those of the generalized twin primes.

2.1 Twin prime classifications

There are two schemes of parametrizations of twin primes which we now describe.

Theorem 2.2. *Let $2D$ be the distance between odd prime numbers p_i, p_f of the pair, D a natural number. Then there are three mutually exclusive classes of generalized twin primes that are parametrized as follows.*

$$I : p_i = 2a - D, p_f = 2a + D, D \text{ odd}; \quad (1)$$

$$II : p_i = 3(2a - 1) - D, p_f = 3(2a - 1) + D, 2 \nmid D, 3 \nmid D; \quad (2)$$

$$III : p_i = 2a + 1 - D, p_f = 2a + 1 + D, 6 \mid D, D \geq 6, \quad (3)$$

where a is the running integer variable. Values of a for which a prime pair of distance $2D$ is reached are unpredictable (called *arithmetic chaos*).

Each of these three classes of this classification [1] contains infinitely many (possibly empty) subsets of prime pairs at given even distance.

Proof. Let us first consider the case of odd D . Then $p_i = 2a - D$ for some positive integer a and, therefore, $p_f = p_i + 2D = 2a + D$. The median $2a$ is the running integer variable of this class *I*.

For even D with D not divisible by 3, let $p_i = 2n + 1 - D$ so that $p_f = 2n + 1 + D$ for an appropriate integer n . Let $p_i \neq 3$, thus excluding a possible first pair with $p_i = 3$ as *special*. Since of three odd natural numbers at distance D from each other one is divisible by 3, the median $2n + 1$ must be so, hence $2n + 1 = 3(2a - 1)$ for an appropriate integer a . Thus, the median $3(2a - 1)$ of the pair $3(2a - 1) \pm D$ is again a linear function of a running integer variable a . These prime number pairs constitute the 2nd class *II*.

This argument is not valid for prime number pairs with $6 \mid D$, but they can obviously be parametrized as $2a + 1 \pm 6d$, $D = 6d$. They comprise the 3rd and last class *III* of generalized twins. Obviously, these three classes are mutually exclusive and complete except for the special cases. \diamond

Example 1.

Ordinary twins $2a \pm 1$ for $a = 2, 3, 6, 9, 15, \dots$ have $D = 1$ and are in class *I*. For $D = 3$, $2a \pm 3$ are twins for $a = 4, 5, 7, 8, 10, \dots$. For $D = 5$, $2a \pm 5$ are twins for $a = 4, 6, 9, 12, 18, \dots$. No twins are ever missed or special in this class *I*, an advantage of this classification.

For $D = 2$, $3(2a - 1) \pm 2$ are twins in class *II* for $a = 2, 3, 4, 7, 8, \dots$.

For $D = 6$, $2a + 1 \pm 6$ are twins in class *III* for $a = 5, 6, 8, 11, 12, \dots$.

Special twins are 5 ± 2 , 7 ± 4 , $11 \pm 8, \dots$.

The *second classification* of generalized twins involves arithmetic progressions of conductor 6 as their regular feature. It is well known that, except for the first pair 3, 5, ordinary twins all have the form $(6m - 1, 6m + 1)$ for some natural number m . They belong to class *I*.

Example 2.

Prime pairs at distance $2D = 4$ are in class *II* and of the form $6m + 1, 6(m+1) - 1$ for $m = 1, 2, 3, \dots$ except for the singlet 3, 7. At distance $2D = 6$ they are in class *I* and have the form $6m - 1, 6(m+1) - 1$ for $m = 1, 2, 3, \dots$ that are intertwined with $6m + 1, 6(m+1) + 1$ for $m = 1, 2, 5, \dots$. At distance $2D = 12$ they are in class *III* and of the form $6m - 1, 6(m+2) - 1$ for $m = 1, 3, 5, \dots$ intertwined with $6m + 1, 6(m+2) + 1$ for $m = 1, 3, 5, \dots$.

In general, the rules governing the form $6m \pm 1, 6m + b$ depend on the arithmetic of D and a making the second classification of generalized twin primes difficult to deal with generally. We now apply Theor. 2.2.

Theorem 2.3. *Let $2D$ be the distance between odd prime numbers p_i, p_f of the pair. Then for class *III*, $D \equiv 0 \pmod{6}$ and $p_{f,i} \equiv 2a + 1 \pmod{6}$. If $a \equiv 0 \pmod{3}$ then $p_{f,i} \equiv 1 \pmod{6}$. If $a \equiv -1 \pmod{3}$ then $p_{f,i} \equiv -1 \pmod{6}$.*

*For class *II*, $D \equiv 2r \pmod{6}$, $r = \pm 1$ yields $p_i \equiv \pm 1 \pmod{6}$, $p_f \equiv \mp 1 \pmod{6}$; $r = \pm 2$ gives $p_i \equiv \mp 1$, $p_f \equiv \pm 1 \pmod{6}$.*

*For class *I*, and $D \equiv 1 + 2r \pmod{6}$, $r = 0, \pm 1$, $a \equiv a_0 \pmod{3}$ the prime pair obeys $p_i \equiv -1 - 2r - 2a_0 \pmod{6}$, $p_f \equiv 1 + 2r + 2a_0 \pmod{6}$. For $r = 0$ and $r = -1$, i.e. $D \equiv \pm 1 \pmod{6}$, $a_0 = 0$. For $r = 1$, i.e. $D \equiv 3 \pmod{6}$, $a \not\equiv 0 \pmod{3}$.*

Proof. For class *III*, $a \equiv 1 \pmod{3}$ is ruled out because then $p_{f,i} \equiv 3 \pmod{6}$. For class *II*, $p_{f,i} = 3(2a - 1) \pm D$, so $D \equiv 2r \pmod{6}$ implies $p_{f,i} \equiv -3 \pm 2r \pmod{6}$ for $r = \pm 1$, etc. For class *I*, and $D \equiv 1 \pmod{6}$ the cases $a \equiv \pm 1 \pmod{3}$ are ruled out because they imply either $3|p_i$ or $3|p_f$, except $a = 2$, i.e. $D = 2$. This also is the case for $D \equiv -1 \pmod{6}$. For $D \equiv 3 \pmod{6}$ the case $a \equiv 0 \pmod{3}$ is obviously ruled out. \diamond

This concludes the classifications of twin primes at constant distances.

2.2 Triplet prime classifications

Generalized triplets have the form $6m \pm 1, 6m + a_1, 6m + a_2$, except for singlet exceptions and this generalizes to prime quadruplets, quintuplets, etc.

Rules for *singlets* or *exceptions* among generalized triplet primes are the following.

Theorem 2.4. (i) *There is at most one generalized prime number triplet with distances $[2D, 2D]$ for $D = 1, 2, 4, 5, \dots$ and $3 \nmid D$.*

(ii) *When the distances are $[2d_1, 2d_2]$ with $3 \mid d_2 - d_1$, and $3 \nmid d_1$, there is at most one triplet $p_i = 3, p_m = 3 + 2d_1, p_f = 3 + 2d_1 + 2d_2$ for appropriate integers d_1, d_2 .*

Proof. (i) Because of three odd numbers in a row one is divisible by 3, 3, 5, 7 is the only triplet at distances $[2, 2]$ and, for the same reason, there is only one triplet 3, 7, 11 at distances $[4, 4]$, one only at distances $[8, 8]$ i.e. 3, 11, 19, at $[10, 10]$ i.e. 3, 13, 23 and in general at distances $[2D, 2D]$ for D not divisible by 3. The argument fails when $3 \mid D$. (ii) Of $p_i, p_m \equiv p_i + 2d_1 \pmod{3}, p_f \equiv p_i + 4d_1 \pmod{3}$ at least one is divisible by 3, which must be p_i . \diamond

Naturally, the question arises: Are there infinitely many such singlets, i.e. exceptional triplets?

Example 3.

At distances $[2, 8]$, the singlet is 3, 5, 13 and at $[8, 2]$, it is 3, 11, 13.

At distances $[2, 4]$, the triplets are $2n - 3, 2n - 1, 2n + 3$ with $n \equiv 1 \pmod{3}$ and at $[4, 2]$, they are $2n - 3, 2n + 1, 2n + 3$ with $n \equiv -1 \pmod{3}$. Writing $n = 3m \pm 1$ for these cases, we obtain $6m - 1, 6m + 1, 6m + 5$ and $6m - 5, 6m - 1, 6m + 1$ for these triplets, respectively. At distances $[2, 4]$, triplets occur for $n = 4, 7, 10, \dots$ i.e. $m = 1, 2, 3, \dots$, while at $[4, 2]$, they are at $n = 5, 8, 20, \dots$ i.e. $m = 2, 3, 7, \dots$. These triplets are in the classes (I, II) and (II, I) , respectively, with I, II denoting symbolically the classes of the first generalized twin prime classification.

At distances $[2, 6]$ the generalized triplets are $2n - 3, 2n - 1, 2n + 5$ with $n \equiv 1 \pmod{3}$ or $6m - 1, 6m + 1, 6m + 7$. The triplet 3, 5, 11 for $n = 3$ is the only exception. For $[6, 2]$ they are $2n - 5, 2n + 1, 2n + 3$ with $n \equiv -1 \pmod{3}$ or $6m - 1, 6m + 5, 6m + 7$. They all are in the class (I, I) .

Applying Theors. 2.2, 2.3 we obtain the following triplet classifications.

Corollary 2.5. (i) *The class (I, I) is made up of the triplets*

$$p_i = 2a - D_1, \quad p_m = 2a + D_1 = 2b - D_2, \quad p_f = 2b + D_2 \quad (4)$$

with odd D_1, D_2 and a, b appropriate integers subject to $b - a = (D_1 + D_2)/2$. Hence

$$p_f = 2a + D_1 + 2D_2, \quad (5)$$

and the prime number pair

$$(p_f, p_i) = 2a + D_2 \pm (D_1 + D_2) \quad (6)$$

is in class II, or special, or III.

(ii) *If*

$$\begin{aligned} D_1 &\equiv 2r_1 + 1 \pmod{6}, r_1 = 0, \pm 1; \quad D_2 \equiv 2r_2 + 1 \pmod{6}, r_2 = 0, \pm 1, \\ a &\equiv a_0 \pmod{3}, a_0 = 0, \pm 1 \end{aligned} \quad (7)$$

then

$$\begin{aligned} p_i &\equiv -2r_1 - 1 + 2a_0 \pmod{6}, \quad p_m \equiv 2r_1 + 1 + 2a_0 \pmod{6}, \\ p_f &\equiv 2a_0 + 3 + 2r_1 + 2r_2 \pmod{6}, \quad D_1 + D_2 \equiv 2(r_1 + r_2) + 2 \pmod{6}. \end{aligned} \quad (8)$$

Now specific cases (ii) can be worked out by substituting values for r_i, a_0 : $D_1 \equiv 1 \pmod{6}$, $a \equiv 0 \pmod{3}$ yield $p_i \equiv -1 \pmod{6}$, $p_m \equiv 1 \pmod{6}$, $p_f \equiv 1 + 2D_2 \pmod{6} \equiv \pm 1 \pmod{6}$ with $D_2 \equiv 1 \pmod{6}$ ruled out, etc. Of course, Example 3 is consistent with this.

Corollary 2.6. (i) *The class (I, II) consists of prime triplets at distances D_1 odd and D_2 even so that*

$$p_i = 2a - D_1, \quad p_m = 2a + D_1 = 3(2b - 1) - D_2, \quad p_f = 3(2b - 1) + D_2 \quad (9)$$

with appropriate a, b subject to $a = 3b - \frac{1}{2}(D_1 + D_2 + 3)$. Hence $p_f = 2a + D_1 + 2D_2$ and the twin

$$(p_f, p_i) = 2a + D_2 \pm (D_1 + D_2) \quad (10)$$

is in class I.

(ii) If

$$\begin{aligned} D_1 &\equiv 2r_1 + 1 \pmod{6}, r_1 = 0, \pm 1; D_2 \equiv 2r_2 \pmod{6}, r_2 = \pm 1, \\ a &\equiv a_0 \pmod{3}, a_0 = 0, \pm 1 \end{aligned} \quad (11)$$

then

$$\begin{aligned} p_i &\equiv -2r_1 - 1 + 2a_0 \pmod{6}, p_m \equiv 2r_1 + 1 + 2a_0 \pmod{6} \\ &\equiv -3 - 2r_2 \pmod{6}, p_f \equiv -3 + 2r_2 \pmod{6}. \end{aligned} \quad (12)$$

Corollary 2.7. *The class (II, II) has D_1 even and D_2 even with both D_i not divisible by 3. The triplets are*

$$\begin{aligned} p_i &= 3(2a - 1) - D_1, p_m = 3(2a - 1) + D_1 = 3(2b - 1) - D_2, \\ p_f &= 3(2b - 1) + D_2 \end{aligned} \quad (13)$$

so that $3(b-a) = \frac{1}{2}(D_1 + D_2)$. Therefore $3|D_1 + D_2$ and $D_1 \equiv -D_2 \pmod{3}$. Since

$$\begin{aligned} p_f &= p_i + 2(D_1 + D_2) = 3(2a - 1) + D_1 + 2D_2, \\ (p_f, p_i) &= 3(2a - 1) + D_2 \pm (D_1 + D_2) \end{aligned} \quad (14)$$

is in class III with $6|D_1 + D_2$.

(ii) If

$$D_i \equiv 2r_i \pmod{6}, r_i = \pm 1 \quad (15)$$

then

$$\begin{aligned} p_i &\equiv -3 - 2r_1 \pmod{6}, p_m \equiv -3 + 2r_1 \pmod{6} \equiv -3 - 2r_2 \pmod{6}, \\ p_f &\equiv -3 + 2r_2 \pmod{6} \equiv p_i \pmod{6}. \end{aligned} \quad (16)$$

Corollary 2.8. *The class (I, III) has the triplets*

$$p_i = 2a - D_1, p_m = 2a + D_1 = 2b + 1 - 6d_2, p_f = 2b + 1 + 6d_2 \quad (17)$$

where $2b + 1 = 2a + D_1 + 6d_2$. So $p_f = 2a + D_1 + 12d_2$ and

$$(p_f, p_i) = 2a + 6d_2 \pm (D_1 + 6d_2) \quad (18)$$

is in class I.

(ii) If

$$D_1 \equiv 1 + 2r_1 \pmod{6}, r_1 = 0, \pm 1; a \equiv a_0 \pmod{3} \quad (19)$$

then

$$\begin{aligned} p_i &\equiv -1 - 2r_1 + 2a_0 \pmod{6}, p_m \equiv 2b + 1 \pmod{6} \equiv \\ &1 + 2r_1 + 2a_0 \pmod{6}, p_f \equiv p_m \pmod{6}, 2b \equiv 2(r_1 + a_0) \pmod{6}. \end{aligned} \quad (20)$$

Corollary 2.9. *Class (II, III) has D_1 even and not divisible by 3 and $D_2 = 6d_2$ with the triplet form*

$$\begin{aligned} p_i &= 3(2a - 1) - D_1, p_m = 3(2a - 1) + D_1 = 2b + 1 - 6d_2, \\ p_f &= 2b + 1 + 6d_2 \end{aligned} \quad (21)$$

so that

$$2b + 1 = 3(2a - 1) + D_1 + 6d_2, p_f = 3(2a - 1) + D_1 + 12d_2. \quad (22)$$

So

$$(p_f, p_i) = 3(2a - 1) + 6d_2 \pm (D_1 + 6d_2) \quad (23)$$

is in class II.

(ii) If

$$D_1 \equiv 2r_1 \pmod{6}, r_1 = \pm 1, a \equiv a_0 \pmod{3}, a_0 = 0, \pm 1 \quad (24)$$

then

$$\begin{aligned} p_i &\equiv -3 - 2r_1 \pmod{6}, p_m \equiv -3 + 2r_1 \pmod{6} \\ &\equiv 2b + 1 \pmod{6}, p_f \equiv p_m \pmod{6}. \end{aligned} \quad (25)$$

Corollary 2.10. *The class (III, III) has the triplet form*

$$p_i = 2a + 1 - 6d_1, p_m = 2a + 1 + 6d_1 = 2b + 1 - 6d_2, 2b + 1 + 6d_2 \quad (26)$$

for appropriate a, b so that

$$b = a + 3(d_1 + d_2), p_f = 2a + 1 + 6(d_1 + 2d_2) \quad (27)$$

and

$$(p_f, p_i) = 2a + 1 + 6d_2 \pm 6(d_1 + d_2) \quad (28)$$

is in class *III*, too.

(ii) If $a \equiv a_0 \pmod{3}$, $b \equiv b_0 \pmod{3}$ then

$$p_i \equiv 2a_0 + 1 \pmod{6} \equiv p_m \pmod{6} \equiv p_f \pmod{6}. \quad (29)$$

The classes (II, I) , (III, I) , (III, II) are handled similarly. Several examples have been given above. These nine classes of generalized prime number triplets are mutually exclusive and complete except for the singlets.

These twin and triplet prime classifications represent regularities that generalize to quadruplet primes which come in 3^3 mutually exclusive classes, quintuplet primes in 3^4 such classes except for singlets, etc.

3 Special twin and triplet primes

Here we consider Mersenne and Fermat twins and triplets.

3.1 Mersenne twins

A simple application of the second classification is the following

Corollary 3.1. *If $2^p - 1$, with an odd prime number p , is a Mersenne prime, then $2^p + 1$ is composite.*

Proof. Since $2^p - 1 \neq 6m - 1$, the pair $2^p \pm 1$ is not a twin prime. \diamond

Of course, it is well known that $3|2^p + 1$ but this requires an algebraic identity for the factorization:

$$a^p + 1 = (a + 1)(a^{p-1} - a^{p-2} \pm \dots + a^2 - a + 1). \quad (30)$$

Let us now consider Mersenne twins with $2^p - 1$ as the *first* prime number of the pair.

Example 4.

The pair $2^p - 1, 2^p + 5$ is a Mersenne pair in class *I* for $p = 3, 5, \dots$, $p \equiv -1 \pmod{3}$ and $p \geq 7$. The qualification is due to the factorization in Lemma 3.2.

The pair $2^p - 1, 2^p + 9$ is a Mersenne pair in class *I* for $p = 2, 3, 5, 7, \dots$

The pair $2^p - 1, 2^p + 3$ is a Mersenne pair in class *II* for $p = 2, 3, 7, \dots$, $p \equiv -1 \pmod{4}$. The restriction is due to Lemma 3.3.

The pair $2^p - 1, 2^p + 11$ is a Mersenne pair in class *III* for $p = 3, 5, 7, \dots$

Lemma 3.2. *If $p \equiv 1 \pmod{3}$ and $p \geq 7$ then $7|2^p + 5$.*

Proof. This follows from the factorization

$$2^p + 2^3 - 2 - 1 = (2^3 - 1)(2^{p-3} + 2^{p-6} + \dots + 2 + 1). \diamond \quad (31)$$

Lemma 3.3. *If $p \equiv 1 \pmod{4}$ then $5|2^p + 3$.*

Proof.

$$2^p + 2^2 - 2 + 1 = (2^2 + 1) \left(\sum_{j=1}^{(p-1)/2} (-1)^{j-1} 2^{p-2j} + 1 \right). \diamond \quad (32)$$

Next we list Mersenne twins with $2^p - 1$ as the *second* prime number of the pair.

Proposition 3.4. *(i) The pair $2^p - 5, 2^p - 1$ is a Mersenne pair in class II for $p = 3$ only. There are no Mersenne prime twins $2^p - 2^{2n+1} - 3, 2^p - 1$; $n = 1, 2, \dots$ except $p = 3, n = 0$. (ii) There are no Mersenne twins $2^p - 3, 2^p - 1$ when $p \equiv -1 \pmod{4}$ except for the pair $5, 7$. (iii) There are no Mersenne twins $2^p - 7, 2^p - 1$ when $p \equiv 1 \pmod{4}$.*

Proof. (i) This holds because $3|2^p - 5$ for $p \geq 3$ which follows from the first factorization

$$\begin{aligned} 2^p - 2 - 2 - 1 &= (2 + 1) \left(\sum_{m=1}^{p-1} (-2)^{p-m} - 1 \right); \\ 2^p - 2^{2n+1} - 2 - 1 &= (2 + 1) \left(\sum_{m=1}^{p-2n-1} (-2)^{p-m} - 1 \right), \\ p &\geq 2n + 3, \quad n = 1, 2, \dots \end{aligned} \quad (33)$$

and the next cases from the second factorization. The case $n = 1$ gives $3|2^p - 11, p \geq 5$.

(ii) is due to the factorization, for $p \equiv -1 \pmod{4}$,

$$2^p - 2^2 + 2 - 1 = (2^2 + 1) \left(\sum_{m=1}^{(p-1)/2} (-1)^{m-1} 2^{p-2m} - 1 \right). \quad (34)$$

(iii) follows from the factorization, for $p \equiv 1 \pmod{4}$,

$$2^p - 2^2 - 2 - 1 = (2^2 + 1) \left(\sum_{m=1}^{(p-1)/2} (-1)^{m-1} 2^{p-2m} - 1 \right). \diamond \quad (35)$$

Example 5.

The pair $2^p - 15, 2^p - 1$ is in class *I*, and $p = 5, 7$ are such cases.

The pair $2^p - 19, 2^p - 1$ is in class *I*, and $p = 5, 7$ are such cases.

The pair $2^p - 13, 2^p - 1$ is in class *III*, and $p = 5, 13$ are such cases.

The pair $2^p - 25, 2^p - 1$ is in class *III*, and $p = 5, 7, 13$ are such cases.

3.2 Fermat twins

Here we consider Fermat prime pairs with the Fermat prime being its first member.

Example 6.

$2^{2^n} + 1, 2^{2^n} + 3$ are twins in class *I* for $n = 0, 1, 2, 4, \dots$, i.e. $2^{n-1} \not\equiv 1 \pmod{3}$, $n > 1$. The qualification is due to (i) in Lemma 3.6.

The pair $2^{2^n} + 1, 2^{2^n} + 7$ is in class *I* and $n = 1, 2, 3$ are such cases.

The pair $2^{2^n} + 1, 2^{2^n} + 13$ is in class *II* and $n = 1, 2, 3$ are such cases.

Corollary 3.5. (i) *The twin prime 3, 7, is the only one of $2^{2^n} + 1, 2^{2^n} + 5$ in class II for $n = 0$.* (ii) *The pair $2^{2^n} + 1, 2^{2^n} + 9$ is in class III and $n = 0, 1$ are the only such cases.*

Proof. (i) follows from the factorization

$$2^{2^n} + 2 + 2 + 1 = (2 + 1) \left(\sum_{j=1}^{2^n-1} (-1)^{j-1} 2^{2^n-j} + 1 \right), \quad n > 0. \quad (36)$$

and (ii) from

$$2^{2^n} + 2^2 + 2^2 + 1 = (2^2 + 1) \left(\sum_{j=1}^{2^{n-1}-1} (-1)^{j-1} 2^{2^n-2j} + 1 \right), \quad n > 0. \quad \diamond \quad (37)$$

Lemma 3.6. (i) *If $2^n \equiv 2 \pmod{3}$, $n > 1$ then $7 \mid 2^{2^n} + 3$.* (ii) *If $2^n \equiv 1 \pmod{3}$ then $7 \mid 2^{2^n} + 5$.*

Proof. (i) follows from the factorization

$$\begin{aligned} 2^{2^n} + 2^3 - 2^2 - 1 &= (2^3 - 1) (2^{2^n-3} + 2^{2^n-6} + \dots + 2^2 + 1), \\ 2^n &\equiv 2 \pmod{3}, \end{aligned} \quad (38)$$

and (ii) from

$$\begin{aligned} 2^{2^n} + 2^3 - 2 - 1 &= (2^3 - 1) (2^{2^n-3} + 2^{2^n-6} + \dots + 2 + 1), \\ 2^n &\equiv 1 \pmod{3}. \quad \diamond \end{aligned} \quad (39)$$

Fermat prime pairs with the Fermat prime being its second member are the following.

Example 7.

The pair $2^{2^n} - 5, 2^{2^n} + 1$ is in class *I* and $n = 2, 3$ are such cases.

The pair $2^{2^n} - 3, 2^{2^n} + 1$ is in class *II* and $n = 2$ is such a case.

The pair $2^{2^n} - 11, 2^{2^n} + 1$ is in class *III* and $n = 2$ is such a case.

Proposition 3.7. *There is no Fermat twin primes of the form $2^{2^n} - 1, 2^{2^n} + 1; 2^{2^n} - 7, 2^{2^n} + 1; 2^{2^n} - 19, 2^{2^n} + 1; 2^{2^n} + 1; 2^{2^n} - 21$.*

Proof. This holds because $5|2^{2^n} - 1, 3|2^{2^n} - 7, 2^{2^n} - 19, 5|2^{2^n} - 21$ which is based on the following factorizations:

$$\begin{aligned}
2^{2^n} - 1 &= (2^2 + 1) \left(\sum_{j=1}^{2^{n-1}-1} (-1)^{j-1} 2^{2^n-2j} - 1 \right), \quad n > 1, \\
2^{2^n} + (-1)^{m+1} 2^{2m} - 2^2 - 1 &= (2^2 + 1) \left(\sum_{j=1}^{2^{n-1}-m} (-1)^{j-1} 2^{2^n-2j} - 1 \right), \\
1 \leq m < 2^{n-1}, \quad n > 1, \\
2^{2^n} - 2^{2m} - 2 - 1 &= (2 + 1) \left(\sum_{j=1}^{2^n-2m} (-1)^{j-1} 2^{2^n-j} - 1 \right), \\
m &= 1, 2, \dots \diamond
\end{aligned} \tag{40}$$

3.3 Mersenne triplets

Here we list prime triplets where the Mersenne prime comes *first*.

Example 8.

$2^p - 1, 2^p + 3, 2^p + 9$ yields triplets for $p = 2, 3 : 3, 7, 13; 7, 11, 13$.

$2^p - 1, 2^p + 5, 2^p + 9$ yields triplets for $p = 3, 5 : 7, 13, 17; 31, 37, 41$ and $2^p - 1, 2^p + 5, 2^p + 11$ yields triplets for $p = 3, 5 : 7, 13, 19; 31, 37, 43$.

$2^p - 1, 2^p + 3, 2^p + 11$ yields triplets for $p = 3, 7 : 7, 11, 19; 127, 131, 139$.

Corollary 3.8. *(i) $2^p - 1, 2^p + 3, 2^p + 7$ for $p = 2$ yields the only such triplet $3, 7, 11$. (ii) $2^p - 1, 2^p + 5, 2^p + 7$ yields no Mersenne triplets, and (iii) $2^p - 1, 2^p + 3, 2^p + 5$ which is in class *II, I* yields no triplets except for $p = 3$, namely $7, 11, 13$.*

Proof. (i) and (ii) follow from the factorization

$$2^p + 2^2 + 2 + 1 = (2 + 1) \left(\sum_{j=1}^{p-2} (-2)^{p-j} + 1 \right), \quad p > 2. \tag{41}$$

(iii) follows from Lemma 3.2 and the factorization

$$\begin{aligned} 2^p + 2^3 - 2^2 - 1 &= (2^3 - 1)(2^{p-3} + 2^{p-6} + \cdots + 2^2 + 1), \\ p &\equiv -1 \pmod{3}. \diamond \end{aligned} \quad (42)$$

Prime triplets where the Mersenne prime comes *last* are the following.

Example 9.

For $p = 3$ the triplet $2^p - 5, 2^p - 3, 2^p - 1$ becomes 3, 5, 7 which is a singlet case.

$2^p - 7, 2^p - 3, 2^p - 1$ yields 23, 29, 31 for $p = 5$.

$2^p - 13, 2^p - 3, 2^p - 1$ yields a triplet for $p = 5 : 19, 29, 31$.

$2^p - 19, 2^p - 15, 2^p - 1$ yields triplets for $p = 5, 7 : 13, 17, 31; 109, 113, 127$.

Corollary 3.9. $2^p - 7, 2^p - 5, 2^p - 1$ yields no Mersenne triplets. $2^p - 1, 2^p + 3, 2^p + 5$ yields no triplets except for $p = 2$.

Proof. This follows from the factorizations (31), (32), (33), (35). \diamond

Corollary 3.10. The triplet $2^p - 3, 2^p - 1, 2^p + 3$ is the only one.

Proof. This follows from the factorization of $5|2^p + 3$ in Eq. (3.3) of Lemma 3.3 for $p \equiv 1 \pmod{4}$ and $5|2^p - 3$ for $p \equiv -1 \pmod{4}$ in Prop. 3.4. \diamond

Finally, prime triplets where the Mersenne prime is in the middle are composed by the twins where the Mersenne prime is last followed by twins where it comes first.

3.4 Fermat triplets

We give a few triplets where the Fermat prime comes *first*.

Example 10.

$2^{2^n} + 1, 2^{2^n} + 3, 2^{2^n} + 9$ yields 3, 5, 11; 5, 7, 13 for $n = 0, 1$.

$2^{2^n} + 1, 2^{2^n} + 5, 2^{2^n} + 11$ yields 3, 7, 13 for $n = 0$.

Corollary 3.11 (i) $2^{2^n} + 1, 2^{2^n} + 3, 2^{2^n} + 5$ for $n = 0$, yields 3, 5, 7 which is the only case. (ii) $2^{2^n} + 1, 2^{2^n} + 5, 2^{2^n} + 9$ for $n = 0$ yields 3, 7, 11 as the only case.

Proof. (i) follows from the factorization (i) and (ii) from (ii) in Prop 3.7. \diamond

We list a few triplets where the Fermat prime comes *last*.

Example 11.

$2^{2^n} - 5, 2^{2^n} - 3, 2^{2^n} + 1$ yields 11, 13, 17 for $n = 2$.

$2^{2^n} - 9, 2^{2^n} - 3, 2^{2^n} + 1$ yields 7, 13, 17 for $n = 2$.

$2^{2^n} - 11, 2^{2^n} - 3, 2^{2^n} + 1$ yields 5, 13, 17 for $n = 2$.

$2^{2^n} - 13, 2^{2^n} - 3, 2^{2^n} + 1$ yields 5, 13, 17 for $n = 2$.

Triplets where the Fermat prime is in the middle are composed by twins where the Fermat prime is last followed by twins where it comes first.

4 Regular prime multiplets

Extensions of the twin and triplet primes at constant distances to quadruplets, quintets, etc. exist but are too numerous to be analyzed systematically here.

4.1 Regular multiplets from quadratic polynomials

We restrict our attention to those with regularly increasing (or decreasing) distances, such as $2N$, $N = 1, 2, \dots$ i.e. $p_1, p_2 = p_1 + 2, p_3 = p_1 + 6, \dots, p_{n+1} = p_1 + n(n+1), \dots, N$ with the sequence of differences $\Delta p_j = p_{j+1} - p_j = 2j$, $j = 1, 2, \dots, N$. Even regular prime triplets and quadruplets are too numerous to be listed. We therefore start with quintuplets in Theor. 4.1 below.

Example 12. There are at least 14 sextets 11, 13, 17, 23, 31, 41; 17, 19, 23, 29, 37, 47; 41, 43, 47, 53, 61, 71; 1277, 1279, 1283, 1289, 1297, 1307; 1607, 1609, 1613, 1619, 1627, 1637; 3527, 3529, 3533, 3539, 3547, 3557; 21557, 21559, 21563, 21569, 21577, 21587; 28277, 28279, 28283, 28289, 28297, 28307; 31247, 31249, 31253, 31259, 31267, 31277; 33617, 33619, 33623, 33629, 33637, 33647; 55661, \dots , 55691; 113147, 113149, 113153, 113159, 113167, 113177; 128981, \dots , 129011; 548831, \dots , 548861; 566537, \dots , 566567; seven septets 11, \dots , 41, 53; 17, \dots , 59; 41, \dots , 83; 1277, \dots , 1319; 21577, \dots , 21599; 28277, \dots , 28319; 113147, \dots , 113189; 128981, \dots , 129023; five octets 11, \dots , 53, 67; 17, \dots , 73; 41, \dots , 97; 21557, \dots , 21599, 21613; 128981, \dots , 129037; three nonets 11, \dots , 67, 83; 17, \dots , 89; 41, \dots , 113 and three decuplets 11, \dots , 83, 101; 17, \dots , 117; 41, \dots , 131.

For $N = 15$ the sequence 17, 19, 23, 29, 37, 47, 59, 73, 89, 107, 127, 149, 173, 199, 227, 257 is a regular 16-plet. The second such 16-plet starts soon after with 41, 43, 47, 53, \dots , $281 = 41 + 15 \cdot 16$. In fact, this one extends much longer and is the first 40-plet ending with $1523, 1601 = 41 + 39 \cdot 40$. There are also many almost-regular prime multiplets where just one member is missing, e.g. $n = 0$.

Naturally, questions arise: Are there infinitely many of these long regular multiplets or even longer ones? Are there other long prime multiplets with-

out the regular structure imposed by a (quadratic) polynomial on its first multiplet which then continues through all of them? Needless to say, the existence of so many regular prime multiplets linking primes with each other in interlocking ways belies the probabilistic independence of prime numbers underlying many conjectures [3],[4] about them. It may not be unreasonable to expect that most multiplets repeat infinitely often. Since they are interlocked their asymptotic distributions are not independent. This suggests that asymptotic laws for prime multiplets differ fundamentally from ordinary prime numbers.

The long regular multiplets of Example 12 are related to Euler's prime number generating polynomials which, in turn, are related to imaginary quadratic number fields over the rationals. There are corresponding polynomials whose values form regular multiplets that are related to real quadratic number fields. Although some of these polynomials have been known for a long time with their large number of prime values as the main focus, their regular distances within the coherent structure of a regular prime number multiplet seem not to have been noted (in print). Except for the multiplet aspects many details below are known and documented in Ref. [2], but some are new.

Theorem 4.1. (i) *The Euler polynomials $E_p(x) = x^2 + x + p$ with the prime numbers $p = 2, 3, 5, 11, 17, 41$ assume prime number values $p + x(x + 1)$ at distances $2(x + 1)$ for $x = 0, 1, \dots, p - 2$ forming a regular $(p - 1)$ -plet.*

(ii) *The polynomials $f_p(x) = 2x^2 + p$ with the primes $p = 3, 5, 11, 29$ assume prime number values at $x = 0, 1, \dots, p - 1$ forming a regular p -plet at distances $2(2x + 1)$.*

(iii) *The polynomials $F_p(x) = 2x^2 + 2x + \frac{1}{2}(p + 1)$ with primes $p = 5, 13, 37 \equiv 1 \pmod{4}$ assume prime values for $x = 0, 1, (p - 3)/2$ which form a regular $(p - 1)/2$ -plet at distances $4(x + 1)$.*

(iv) *The polynomials G_p , with primes p, q ,*

$$\begin{aligned} G_p(x) &= px^2 + px + \frac{1}{4}(p + q), \quad p < q, pq \equiv 3 \pmod{4}, \\ (p, q) &= (3, 5), (3, 17), (3, 41), (3, 89), (5, 7), (5, 23), (5, 47), (7, 13), (7, 61), \\ &\quad (11, 17), (13, 31) \end{aligned} \tag{43}$$

assume prime values for $x = 0, 1, \dots, \frac{1}{4}(p + q) - 2$. which form regular multiplets at distances $2p(x + 1)$ that are independent of the prime q .

(v) *Polynomials corresponding to real quadratic number fields are $g_d(x) = -x^2 + x + \frac{1}{4}(d - 1)$ with $d > 0, d \equiv 1 \pmod{4}$ and square-free. Their values*

are prime numbers for $x = 2, 3, \dots < \frac{1}{2}\sqrt{d-1}$ forming regular multiplets at distances $2p(x+1)$. Relevant values are $d = 37, 53, 77, 101, 173, 197, 293, 437, 677$.

(vi) Quintets, sextets, septets, octets, nonets and decuplets are generated by the following polynomials

$$\begin{aligned}
Q_p(x) &= x^2 + x + p, \quad x = 0, \dots, 4; \quad p = 11, 17, 41, 347, 641, 1427, 4001, \\
&\quad 4637, 4931, 19421, 22271, 23471, 26711, 27941, 28277, \\
&\quad 31247, 32057, 33617; 113147 \dots; \\
SX_p(x) &= x^2 + x + p, \quad x = 0, \dots, 5; \quad p = 11, 17, 41, 1277, 1607, 3527, \\
&\quad 28277, 31247, 33617, 55661, 113147, 128981, 548831, 566537, \dots; \\
S_p(x) &= x^2 + x + p, \quad x = 0, \dots, 6; \quad p = 11, 17, 41, 1277, 28277, \\
&\quad 113147, 128981, \dots; \\
O_p(x) &= x^2 + x + p, \quad x = 0, \dots, 7; \quad p = 11, 17, 41, 128981, \dots; \\
N_p(x) &= x^2 + x + p, \quad x = 0, \dots, 8; \quad p = 11, 17, 41, \dots; \\
D_p(x) &= x^2 + x + p, \quad x = 0, \dots, 9; \quad p = 11, 17, 41, \dots
\end{aligned} \tag{44}$$

Proof. (i) It is long known [2] that for the listed primes p the values $E_p(x) = x^2 + x + p$ at $x = 0, 1, 2, \dots, p-2$ are prime numbers. Since

$$E_p(x+1) - E_p(x) = (x+1)^2 - x^2 + 1 = 2(x+1), \tag{45}$$

the primes $p + x(x+1)$ form a regular $(p-1)$ -plet at distances $2(x+1)$ from each other for $x = 0, 1, 2, \dots, p-2$.

(ii) Since $f_p(x+1) - f_p(x) = 2(2x+1)$, the primes $p+2x^2$, $x = 0, 1, \dots, p-1$ form a regular p -plet at distances $2(2x+1)$.

(iii) Since $F_p(x+1) - F_p(x) = 4(x+1)$ the primes $2x(x+1) + (p+1)/2$ form a regular $(p-1)/2$ -plet at distances $4(x+1)$.

(iv) Since $G_p(x+1) - G_p(x) = 2p(x+1)$ the multiplet structure is clear.

(v) For g_d , this follows from $g_d(x+1) - g_d(x) = -2x$. These are multiplets with linearly decreasing distances.

(vi) These multiplets may be verified by a table of prime numbers or symbolic-math software. \diamond

There are many more recently found polynomials in the literature [2] which also form regular prime multiplets.

4.2 Optimal quadratic polynomials

This subject has a long history [5] with a rather unsystematic record. It is well known that, if $P(x) = \sum_{j=0}^n a_j x^j$ is a non-constant polynomial of degree $n \geq 1$ with integral coefficients a_j , $|a_0| = p_0$ a prime number, then $P(x)$ can assume prime values at most for $x = 0, 1, \dots, p_0 - 1$ because $p_0 | P(p_0)$.

Definition 4.2. The polynomial $P(x) = \sum_{j=0}^n a_j x^j$ is called *optimal* if $|P(j)| = p_j$ is prime for $j = 0, 1, \dots, p_0 - 1$, forming a regular p_0 -plet.

Prime number values have to be successive, but they may repeat and be negative. This is often caused by negative coefficients in a polynomial.

As $E_p(p-1) = p^2$, the Euler polynomials in (i) of Theor. 4.1 for $p = 2, 3, 5, 11, 17, 41$ are one prime value short of being optimal. But Legendre's quadratic polynomials [5] $f_p(x) = 2x^2 + p$ for $p = 3, 5, 11, 29$ in (ii) of Theor. 4.1 are optimal.

Since $E_p(-x) = E_p(x-1)$, Euler polynomials give repeating prime multiplets when they are considered for positive and negative argument. For more general polynomials this is not the case. Upon shifting the argument, the modified Euler polynomials $E_p(x-1) = x(x-1) + p$ do become optimal, but they repeat the initial prime value. More generally, the identity

$$E_q(x-n) = x^2 + (1-2n)x + E_q(n-1) \quad (46)$$

for prime numbers q , $p = E_q(n-1)$ and nonnegative integer n leads to many new repeating polynomials with more numerous prime values. For $q = 41$ and $n = 2, 3, \dots, 40$ none of these polynomials is optimal (cf. Eq. (61)) in Cor. 4.5 e.g.), though, including Escott's for $n = 40$ [2], [5].

Definition 4.3. The polynomial $P(x)$ is *bi-optimal* if $|P(j)|$ are prime for $1-p_0 \leq j \leq p_0-1$, forming a $(2p_0-1)$ -plet.

This corresponds to combining $P(j)$ for $j = 0, 1, \dots, p_0-1$ and $P(-j)$ for $j = 1, \dots, p_0-1$ into one prime number multiplet. For polynomials of odd degree the starting value $x = 0$ is appropriate. For polynomials of even degree it is perhaps more natural to include negative arguments as well.

We now address a question raised by Legendre's $f_p(x)$ and Euler's modified polynomials: Are there optimal quadratic polynomials for the missing-link primes $p = 7, 13, 19, 23, 31, 37$?

For $p = 7$, $Q_2(x) = 2x(x-1) + 7$ is optimal, if repeating, generating a regular prime number septet with distances $\Delta Q_2(x) = 4x$, and it is almost bi-optimal because $Q_2(-x) = Q_2(x+1)$.

For $p = 19$, $Q_7(x) = 2x(x - 1) + 19$ [5] is not only optimal but almost bi-optimal in view of $Q_7(-x) = Q_7(x + 1)$, generating a regular 36-plet with the same distance law as Q_2 .

For $p = 23$, $Q_8(x) = 3x(x - 1) + 23$ [5] is not only optimal but almost bi-optimal forming a 42-plet because $Q_8(-x) = Q_8(x + 1)$.

Proposition 4.4. (i) *There is an infinity of optimal quadratic polynomials for $p_0 = 2$:*

$$Q_1(x) = x^2 + (p_1 - 3)x + 2, \quad (47)$$

where p_1 is prime. If $|p_1 - 6|$ is prime then $Q_1(x)$ is bi-optimal.

(ii) *There are at least three quadratic polynomials for $p_0 = 13$:*

$$\begin{aligned} Q_3(x) &= x^2 + 27x + 13, & Q_4(x) &= x^2 - 3x + 13, \\ Q_5(x) &= 2x^2 - 4x + 13, \end{aligned} \quad (48)$$

that form 12-plets, one prime value short of optimal. Hence

$$Q_3(x - 1) = x^2 + 25x - 13 \quad (49)$$

is optimal. $Q_5(-x)$ forms a decuplet making $Q_5(x)$ almost bi-optimal. There is another polynomial,

$$Q_6(x) = 2x^2 + 26x + 13, \quad (50)$$

that forms a decuplet.

Proof. (i) follows from $Q_1(0) = 2$, $Q_1(1) = p_1$ and $Q_1(-1) = 6 - p_1$.

(ii) For monic optimal polynomials let

$$Q(x) = x^2 + bx + p_0, \quad Q(1) = p_1 = p_0 + b + 1, \quad Q(2) = p_2 = p_0 + 4 + 2b, \quad (51)$$

where p_0, p_1, p_2 are prime numbers. Then

$$b = p_1 - p_0 - 1, \quad Q = x^2 + (p_1 - p_0 - 1)x + p_0, \quad p_2 = 2p_1 - p_0 + 2. \quad (52)$$

If $Q(x) = 2x^2 + bx + p_0$, then

$$Q(1) = p_1 = p_0 + b + 2, \quad Q(2) = p_2 = p_0 + 2b + 8. \quad (53)$$

(ii) For $p_0 = 13$ and $p_1 = 41$ we obtain Q_3 from Eq. (52) and the 12-plet

$$\begin{aligned} Q_3(1) &= 41, & Q_3(2) &= 71, & Q_3(3) &= 103, & Q_3(4) &= 137, & Q_3(5) &= 173, \\ Q_3(6) &= 211, & Q_3(7) &= 251, & Q_3(8) &= 293, & Q_3(9) &= 337, & Q_3(10) &= 383, \\ Q_3(11) &= 431, \end{aligned} \quad (54)$$

one prime value short of optimal. Q_3 is ascending and non-repeating.

For $p_0 = 13$, $p_1 = 11$ we get the monic Q_4 , again with a 12-plet of prime values

$$\begin{aligned} Q_4(1) &= 11, Q_4(2) = 11, Q_4(3) = 13, Q_4(4) = 17, Q_4(5) = 23, \\ Q_4(6) &= 31, Q_4(7) = 41, Q_4(8) = 53, Q_4(9) = 67, Q_4(10) = 83, \\ Q_4(11) &= 101, \end{aligned} \tag{55}$$

which is one prime value short of optimal. It is repeating and non-ascending because $Q_4(1) = 11 = Q_4(2)$, $Q_4(3) = 13 = Q_4(0)$.

For $p_1 = 11$ we get Q_5 from Eq. (53) and the 12-plet

$$\begin{aligned} Q_5(1) &= 11, Q_5(2) = 13, Q_5(3) = 19, Q_5(4) = 29, Q_5(5) = 43, \\ Q_5(6) &= 61, Q_5(7) = 83, Q_5(8) = 109, Q_5(9) = 139, Q_5(10) = 173, \\ Q_5(11) &= 211, \end{aligned} \tag{56}$$

which is one prime value short of optimal. It is obviously repeating and non-ascending. Since

$$Q_5(-x) = 2x^2 + 4x + 13 = Q_5(x + 2), \tag{57}$$

it generates a decuplet. Therefore, $Q_5(x)$ is almost bi-optimal.

For $p_1 = 41$ we get Q_6 from Eq. (53) and the decuplet

$$\begin{aligned} Q_6(2) &= 73, Q_6(3) = 109, Q_6(4) = 149, Q_6(5) = 193, Q_6(6) = 241, \\ Q_6(7) &= 293, Q_6(8) = 349, Q_6(9) = 409. \diamond \end{aligned} \tag{58}$$

Corollary 4.5. *The polynomial*

$$Q_9(x) = x^2 + 3x + 19 \tag{59}$$

forms a 15-plet for $x = 0, \dots, 14$,

$$Q_{10}(x) = x^2 - x + 11, Q_{11}(x) = 2x^2 + 22x - 11 \tag{60}$$

are optimal polynomials for 11, and

$$Q_{12}(x) = 2x^2 - 4x + 31, Q_{13}(x) = x^2 - 3x + 43 \tag{61}$$

form 30- and 42-plets, respectively, one prime value short of optimal.

Proof. This follows from Prop. 4.4 in conjunction with

$$\begin{aligned} Q_9(x-1) &= x^2 + x + 17 = E_{17}(x), \quad Q_{10}(x) = E_{11}(x-1), \\ Q_{11}(x) &= Q_6(x-1), \quad Q_{12}(x) = f_{29}(x-1), \\ Q_{13}(x) &= E_{41}(x-2), \end{aligned} \tag{62}$$

respectively. \diamond

We leave open the questions: Are there primes for which there is no optimal quadratic polynomial? Are $p = 31, 37$ such cases?

4.3 Optimal cubic polynomials

We now investigate cubic polynomials for the prime numbers $p_0 = 2, 3, 5, 7, 11, 13$.

Theorem 4.6. (i) *There is an infinity of optimal cubic polynomials for the prime numbers 2, 3. For $p_0 = 2$:*

$$C_1(x) = x^3 + mx^2 + (p_1 - 3 - m)x + 2, \quad C_1(1) = p_1, \tag{63}$$

where m is an arbitrary integer and p_1 a prime number. If $|4 + 2m - p_1|$ is prime then $C_1(x)$ is bi-optimal. For $p_0 = 3$:

$$\begin{aligned} C_2(x) &= x^3 + \left[\frac{1}{2}(p_2 - 3) - p_1\right]x^2 + \left[2p_1 - \frac{1}{2}(p_2 + 5)\right]x + 3, \\ C_2(1) &= p_1, \quad C_2(2) = p_2, \end{aligned} \tag{64}$$

where p_1, p_2 are prime and p_2 odd. If $|p_2 - 3p_1 + 3|, |3p_2 - 8p_1 - 6|$ are prime then $C_2(x)$ is bi-optimal.

(ii) *There are optimal polynomials for $p_0 = 5$:*

$$C_3(x) = x^3 - x^2 + 2x + 5, \quad C_4(x) = 2x^3 + 4x^2 - 4x + 5. \tag{65}$$

$p_0 = 7$:

$$C_5(x) = x^3 - x^2 + 6x + 7, \tag{66}$$

$p_0 = 11$:

$$C_6(x) = x^3 + 5x^2 + 2x + 11, \quad C_7(x) = x^3 - 4x^2 + 5x + 11, \tag{67}$$

$p_0 = 13 :$

$$C_8(x) = x^3 - 5x^2 + 8x + 13. \quad (68)$$

Proof. (i) $C_1(1) = p_1$ is readily verified for $p_0 = 2$ and $C_2(1) = p_1$, $C_2(2) = p_2$ for $p_0 = 3$. Note that $C_1(-1) = 4 + 2m - p_1$, $C_2(-1) = p_2 - 3p_1 + 3$, $C_2(-2) = 3p_2 - 8p_1 - 6$.

(ii) Let $C(x) = ax^3 + bx^2 + cx + p_0$. Then

$$\begin{aligned} p_1 &= a + b + c + p_0, \quad p_2 = 8a + 4b + 2c + p_0, \\ p_3 &= 27a + 9b + 3c + p_0. \end{aligned} \quad (69)$$

(ii) For $p_0 = 5$, choosing $a = 1, p_1 = 7, p_2 = 13$ yields $b + c = 1$, $2b + c = 0$ and therefore $C_3(x)$ yielding the following additional prime values

$$C_3(3) = 29, \quad C_3(4) = 61, \quad (70)$$

forming altogether a quintet.

For $a = 2, p_0 = 5, p_1 = 7, p_2 = 29$ we get $c = -b$ and $b = 4$. The polynomial C_4 yields the additional prime values

$$C_4(3) = 83, \quad C_4(4) = 181, \quad (71)$$

forming a quintuplet. For $p_0 = 7, p_1 = 13, p_2 = 23$ we get $b = -1, c = 6$ and C_5 yields the additional prime values

$$C_5(3) = 43, \quad C_5(4) = 79, \quad C_5(5) = 137, \quad C_5(6) = 223, \quad (72)$$

a septet and optimal. For $p_0 = 11, p_1 = 19, p_2 = 43$ we get $b = 5, c = 2$ and C_6 yields the prime values

$$\begin{aligned} C_6(3) &= 89, \quad C_6(4) = 163, \quad C_6(5) = 311, \quad C_6(6) = 419, \quad C_6(7) = 613, \\ C_6(8) &= 859, \quad C_6(9) = 1163, \quad C_6(10) = 1531, \end{aligned} \quad (73)$$

forming an optimal 11-plet. It may be verified that C_7 forms an 11-plet also. $C_8(x)$ generates the optimal 13-plet

$$\begin{aligned} C_8(0) &= 13, \quad C_8(1) = 17, \quad C_8(2) = 17, \quad C_8(3) = 19, \quad C_8(4) = 29, \\ C_8(5) &= 53, \quad C_8(6) = 97, \quad C_8(7) = 167, \quad C_8(8) = 269, \quad C_8(9) = 409, \\ C_8(10) &= 593, \quad C_8(11) = 827, \quad C_8(12) = 1117. \quad \diamond \end{aligned} \quad (74)$$

Concluding we ask: Are there optimal cubic polynomials for all prime numbers $p > 13$?

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